JOURNAL OF APPROXIMATION THEORY 13, 73–83 (1975)

Spline Solutions to L^1 Extremal Problems in One and Several Variables

S. D. FISHER AND J. W. JEROME*

Department of Mathematics, Northwestern University, Evanston, Illinois 60201

Communicated by P. L. Butzer

DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION OF HIS SIXTY-FIFTH BIRTHDAY

1. INTRODUCTION

Let $a = x_0 < x_1 < \cdots < x_m = b$ be a fixed partition of the closed interval $[a, b]$ and let U be the closed flat in the Sobolev space $W^{n,1}(a, b)$, $1 \leqslant n \leqslant m + 1$, defined by the interpolation to specified values r_i at the points x_i, $i = 0, 1,..., m$. If

$$\alpha = \inf\{\| D^n f \|_{L^1(a,b)} : f \in U\}, \tag{1}$$

then the minimization problem (1) does not, in general, have an interpolating solution in the class $W^{n,1}(a, b)$. More generally, if U is a closed flat in a Banach space X, and R is a continuous linear mapping of X onto Y with finite-dimensional null space, then it is possible that $\inf\{\| Ru \| Y: u \in U\}$ is not attained in U; for example, if Y is not reflexive. In his recent paper [6], Holmes has discussed, with a both a literature survey and new results, the technique of embedding such optimization problems in dual spaces. By considering the extended problem of minimizing $\| R^{**}\varphi \|_{Y^{**}}$ over the flat JU in X^{**}, where J denotes the natural injection of X into X^{**}, Holmes has shown that a solution φ exists in JU, achieving the same norm extremal value as in the original problem under a natural poisedness hypothesis. Thus, the problem has a solution in the sense of Ioffe and Tihomirov [7].

In this paper we shall discuss concrete ways in which such problems can be extended to possess natural solutions. In particular, for the problem (1), we expand the class $W^{n,1}(a, b)$ to include functions whose nth derivatives

* Research sponsored by National Science Foundation Grant GP 32116.

are measures. The extended problem has a solution s such that Var $D^{n-1}s$ coincides with the extremal value of (1). s has the property that $s \in C^{n-2}[a, b]$ and is a spline function of degree $n - 1$. This occurs because the measure $D^n s$ has minimal support, concentrated at $m + 1$ or fewer points, as we shall show. Our results are very general and apply in several variables as well as one variable, with appropriate operators. Also, more generally, U may be a slab defined by arbitrary continuous linear inequality constraints in (1) and D^n may be replaced by more general differential operators.

The plan of the paper is as follows. In Section 1 we present the basic existence theorem concerning extremal measures in general context. This convex set of extremal measures, all of which have minimal total variation, is shown to satisfy the hypotheses of the Krein–Milman theorem so that there is at least one extremal measure, whose support is shown to be a finite point set. Univariate and multivariate applications of this result are given in Section 2 together with a linear programming application. In Section 3 we examine the approximation properties of the solutions of the extended problem (1): for functions f in $W^{n,1}(a, b)$ with $\| D^n f \|_{L^1[a,b]} \leq 1$ the order of L^p approximation, for $1 \leq p \leq \infty$, is shown to be $O(h^{n-1})$, where h denotes the maximum mesh length. For certain cases, this is shown to be best possible in $L^1(a, b)$.

We make some final remarks about the solution of (1) and its generalizations. First there is, of course, no claim made regarding uniqueness. Second, it is of some interest to compare the analytical character of the spline solutions of degree $n - 1$ with that of solutions of comparable L^p problems for $1 < p \leq \infty$. For $p = 2$ it has long been known that the unique solution is a spline function of degree $2n - 1$. For $p = \infty$, there exist spline solutions of degree n, first shown by Favard [14]; cf. also, Smith [12]. A general treatment both in one variable [2] and in several variables [3] which displays bang-bang phenomena was given by the authors. For $1 < p < \infty$, $p \neq 2$, the unique solutions satisfy locally a nonlinear differential Euler equation. This problem was completely solved by Golomb [5] and, in special cases, by Mangasarian and Schumaker [9].

1. A Theorem on Constrained Extremal Measures with Minimal Support

Let X be a compact metric space, $C(X)$ the Banach space of real-valued continuous functions on X in the supremum norm and $M(X)$ the dual Banach space of real (finite, regular) Baire measures on X in the total variation norm. Let N be a finite dimensional subspace of $C(X)$ and let H be the direct sum of $M(X)$ and N.

THEOREM 1. *Let $L_0, \ldots, L_m$ be linearly independent linear functionals on H of the form*

$$L_i(\lambda, P) = \int_X F_i d\lambda + l_i(P),$$

where F_i is continuous on X, $i = 0, \ldots, m$ and $\{l_i\}_{i=0}^m$ is a set of linear functionals defined on N such that $P = 0$ if $P \in N$ and $l_i(P) = 0$ for each $i = 0, \ldots, m$. Let $I_0, \ldots, I_m$ be compact intervals in $\mathbf{R}$, each possibly consisting of a single point, and let

$$U = \{(\lambda, P) \in H: L_i(\lambda, P) \in I_i, i = 0, 1, \ldots, m\}.$$

Set

$$\alpha = \inf\{\| \lambda \|: (\lambda, P) \in U\}. \tag{1.1}$$

Then U contains at least one pair (λ, P) for which $\| \lambda \| = \alpha$; the set S of such pairs is convex and compact in the weak topology. The extreme points of S are all of the form $(\sum_0^r c_j \delta_j, P)$ where $r \leqslant m$, δ_j is the unit point mass at the $t_j \in X$ for $j = 0, \ldots, r$ and $\sum_0^r | c_j | = \alpha$.*

Proof. Let $\{g_\nu = (\lambda_\nu, P_\nu)\}$ be a sequence in U with $\| \lambda_\nu \| \searrow \alpha$. Since $L_i(g_\nu) \in I_i$ for all ν and since $\int_X F_i \, d\lambda_\nu$ is also uniformly bounded for all ν and i, we find that

$$| l_i(P_\nu)| \leqslant C, \qquad i = 0, \ldots, m \quad \text{and} \quad \nu = 1, 2, \ldots . \tag{1.2}$$

Hence, by the completeness of $\{l_i\}_0^m$ it follows from (1.2) that the sequence P_ν is bounded in norm. Thus, there is a subsequence of $\{P_\nu\}$, denoted $\{P_{\nu_j}\}$, and a $P_0 \in N$ with $l_i(P_{\nu_j}) \to l_i(P_0)$ as $j \to \infty$ for $i = 0, \ldots, m$. Likewise, the measures $\{\lambda_{\nu_j}\}$ have a weak* accumulation point λ_0 with $\| \lambda_0 \| \leqslant \alpha$. It is easy to check that $(\lambda_0, P_0) \in U$ and hence $\| \lambda_0 \| = \alpha$. Thus we have shown that S is nonempty. The convexity of S follows from the convexity of U and the definition of α. Now the convex set T of measures determined as the set of first components of S is clearly weak* closed in $M(X)$ since it is bounded and contains all its weak* sequential limit points (since X is compact metric, $C(X)$ is separable [4, p. 276, Theorem 14–9.15] and hence the closed unit ball of $M(X)$ is metrizable in the weak-* topology [1, p. 426]). Now by the Krein–Milman theorem let (λ, P) be any extreme point of S and suppose there are $m + 2$ disjoint Baire sets $E_0, \ldots, E_{m+1}$ in X which have positive λ-measure. Let λ_i be the restriction of λ to E_i for $i = 0, \ldots, m + 1$ and let v_i be the vector in $(m + 1)$-space whose jth coordinate is $L_j(\lambda_i, P)$, $j = 0, \ldots, m$ and $i = 0, \ldots, m + 1$. The vectors $v_0, \ldots, v_{m+1}$ must be linearly dependent in $\mathbf{R}^{m+1}$ and hence there are scalars $a_0, \ldots, a_{m+1}$, not all zero, with $\sum_0^{m+1} a_i v_i = 0$. Let

$\mu = \sum_0^{m+1} a_i\lambda_i$ so that μ is not the zero measure. Then we have, for $0 < \epsilon$ sufficiently small,

$$\|\lambda + \epsilon\mu\| = \|\lambda\|_{X-\bigcup_j E_j} + \sum_0^{m+1} (1 + \epsilon a_i)\|\lambda_i\| = \alpha + \epsilon \sum_0^{m+1} a_i \|\lambda_i\|.$$

Now, if $\sum_0^{m+1} a_i \|\lambda_i\| \neq 0$, then some choice of ϵ gives a pair $(\lambda + \epsilon\mu, P)$ in U with $\|\lambda + \epsilon\mu\| < \alpha$, a contradiction. Hence, $\sum_0^{m+1} a_i \|\lambda_i\| = 0$ so that each of $(\lambda + \epsilon\mu, P)$, $(\lambda - \epsilon\mu, P)$ lies in S. The convex representation

$$(\lambda, P) = \tfrac{1}{2}(\lambda - \epsilon\mu, P) + \tfrac{1}{2}(\lambda + \epsilon\mu, P)$$

then contradicts the choice of (λ, P) as an extreme point of S. It follows that there are at most $m + 1$ disjoint subsets of X of positive λ measure and the theorem follows.

2. Applications

(I) *Univariate Generalized Splines*

Let $I = [a, b]$ be a closed interval in $\mathbf{R}$ and let L be a nonsingular linear differential operator of order n on I of the form

$$L = D^n - \sum_{j=0}^{n-1} a_j D^j, \qquad n \geqslant 2,$$

where $a_j \in C(I)$, $j = 0,..., n-1$. If $f \in W^{n,1}(I)$ then the representation

$$f(x) = P(x) + \int_a^b \hat{\theta}(x, \xi)\, Lf(\xi)\, d\xi, \qquad a \leqslant x \leqslant b, \tag{2.1}$$

holds, where P in the null space N_L of L is defined by $D^jP(a) = D^jf(a)$, $0 \leqslant j \leqslant n-1$ and where the function $\theta(\cdot, \xi) \in N_L$ is defined for each $\xi \in [a, b]$ by

$$[D_x{}^j\theta(x, \xi)]_{x=\xi} = \delta_{j,n-1}, \qquad 0 \leqslant j \leqslant n-1,$$

and $\hat{\theta}$ is given by

$$\hat{\theta}(x, \xi) = \begin{cases} \theta(x, \xi) & \text{if } a \leqslant \xi \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}.$$

Here $W^{n,1}(I)$ is the real Sobolev class of $f \in C^{n-1}(I)$ such that $D^{n-1}f$ is absolutely continuous and $D^nf \in L^1(I)$. In this application we shall have need of the larger class $Q^n(I) = \{f: Lf \in M(I)\}$ where Lf is taken in the weak sense and $M(I)$ is the space of real Baire measures on I. Equivalently,

$$f(x) = P(x) + \int_a^b \hat{\theta}(x, \xi)\, d\mu(\xi), \tag{2.2}$$

for $\mu \in M(I)$ and $P \in N_L$ if $f \in Q^n(I)$. We remark that

$$\| Lf \|_{L^1(I)} = \| Lf \|_{M(I)} \quad \text{if} \quad f \in W^{n,1}(I).$$

Now let $\Lambda = \{\lambda_0, ..., \lambda_m\}$ be a linearly independent set of linear functionals on $C^{n-2}(I)$ for which the Peano representations

$$\lambda f = \lambda P + \int_a^b \lambda\hat{\theta}(., \xi)\, Lf(\xi)\, d\xi, \tag{2.3}$$

hold for every $\lambda \in \Lambda$ and $f \in W^{n,1}(I)$ with $\{\lambda\hat{\theta}(\cdot, \xi)\}_{\lambda\in\Lambda}$ a linearly independent set of continuous functions in ξ on I where P is given by (2.1). Since $L^1(I)$ is weak* dense in $M(I)$, (2.3) holds over the larger class $Q^n(I)$. Let $I_0, ..., I_m$ be compact intervals in $\mathbf{R}$, which are possibly single points, and let

$$U = \{f \in W^{n,1}(I): \lambda_i(f) \in I_i,\ i = 0, ..., m\}.$$

THEOREM 2.1. *Let $\alpha = \inf\{\| Lf \|_{L^1[I]}: f \in U\}$. Then there are distinct points $t_0 < \cdots < t_r$ in I, $r \leqslant m$, real numbers $a_0, ..., a_r$ and a function $P \in N_L$ such that the generalized spline function*

$$s(x) = \sum_{j=0}^{r} a_j\hat{\theta}(x, t_j) + P(x),$$

satisfies $\lambda_i s \in I_i$, $i = 0, ..., m$ and $\sum_{j=0}^{r} | a_j | = \alpha$, i.e., $\| Ls \|_{M[I]} = \alpha$. s solves the extended minimization problem $\alpha = \| Ls \|_{M[I]} = \inf\{\| Lf \|_{M[I]}: f \in \tilde{U}\}$. Here $\tilde{U} \supset U$ consists of all $f \in Q^n(I)$ satisfying $\lambda_i f \in I_i$, $i = 0, ..., m$. Finally,

$$Ls = \sum a_j \delta(., t_j), \tag{2.4}$$

where $\delta(\cdot, t)$ represents the Dirac delta functional at t.

Proof. $W^{n,1}(I)$ is algebraically isomorphic to $L^1(I) \oplus N_L$ under (2.1). Now $L^1(I) \oplus N_L$ is weak* dense in $M(I) \oplus N_L$ so that

$$\tilde{\alpha} = \inf\{\| Lf \|_{M[I]}: f \in \tilde{U}\} = \alpha.$$

Indeed, if $\mu_0 = Lf_0$ is chosen so that $\| \mu_0 \| = \tilde{\alpha}$, $f_0 \in \tilde{U}$, choose $\Psi_\nu \to^* \mu_0$ such that $\Psi_\nu \in L^1(I)$, $\nu = 1, 2,...$ and $\| \Psi_\nu \|_{L^1[I]} \leqslant \| \mu_0 \|_{M[I]}$. We show how to adjust Ψ_ν slightly so as to obtain an element which, upon integration, is in U.

Thus, we consider the map $K: L^1(I) \to \mathbf{R}^{m+1}$ given by

$$Kg = \left(\int_I F_0 g\,dx, \ldots, \int_I F_m g\,dx\right),$$

where $F_i(\xi) = \lambda_i \hat{\theta}(\cdot, \xi)$, $i = 0, \ldots, m$. K is clearly onto by the linear independence of the F_i and the quotient space $L^1(I)/\ker K$ is algebraically and topologically isomorphic to $\mathbf{R}^{m+1}$. Thus, there exist sequences $\epsilon_\nu \to 0$ and $\{\varphi_\nu\} \subset L^1(I)$ such that $\| \varphi_\nu \| \leqslant \epsilon_\nu$ and $\varphi_\nu + \Psi_\nu \in LU$, $\nu = 1, \ldots$.

Since

$$\alpha \leqslant \liminf_{\nu \to \infty} \| \Psi_\nu + \varphi_\nu \|_{L^1(I)} = \liminf_{\nu \to \infty} \| \Psi_\nu \|_{L^1(I)} \leqslant \| \mu_0 \|_{M(I)} = \tilde{\alpha},$$

it follows that $\alpha = \tilde{\alpha}$. The result now follows from Theorem 1.1 and the representation (2.2).

COROLLARY 2.2. *Consider the extremal problem* (1) *of the introduction. Then there is a polynomial spline function of degree* $n - 1$ *in* $C^{n-2}[a, b]$ *with at most* $m + 1$ *knots in* (a, b) *satisfying* $\operatorname{Var} D^{n-1}s = \alpha$ *and* $s(x_i) = r_i$, $i = 0, \ldots, m$. Here $n \geqslant 2$.

(II) *Multivariate Generalized Spline Functions*

Let Ω be a bounded domain in $\mathbf{R}^l$, $l \geqslant 2$, and let L be a linear differential operator of order n. Our fundamental assumption concerning L is that the mapping

$$L: W^{n,1}(\Omega) \to L^1(\Omega),$$

is continuous and surjective. Here $W^{n,1}(\Omega)$ is the Sobolev space of functions f with distribution derivatives $D^{\alpha'} f \in L^1(\Omega)$, $| \alpha' | \leqslant n$, with norm

$$\| f \|_{W^{n,1}(\Omega)} = \sum_{|\alpha'| \leqslant n} \| D^{\alpha'} f \|_{L^1(\Omega)} . \tag{2.5}$$

We shall further assume that there is a closed linear subspace F of $W^{n,1}(\Omega)$ such that the restriction of L to F admits a unique inverse representation of the form

$$f(x) = \int_\Omega G(x, \xi)\, Lf(\xi)\, d\xi, \qquad f \in F, \tag{2.6}$$

where we explicitly assume that $G(x, \cdot) \in C(\bar{\Omega})$ for each $x \in \Omega$.

We have in mind, of course, the specific application where L is a uniformly elliptic operator of even order $n = 2k$, in which case, if $n > l$, the above hypotheses are satisfied for sufficiently smooth boundary $\partial\Omega$, for the choice $F = W_o^{2k,1}(\Omega) \cap W^{k,1}(\Omega)$ and $L + \lambda I$, if λ is sufficiently large. Here $W_o^{k,1}(\Omega)$ is the completion, in the norm (2.5), of the $C^\infty(\Omega)$ functions with compact support in Ω.

Let $\Lambda = \{\lambda_0, ..., \lambda_m\}$ be any linearly independent set of linear functionals on F such that the Peano representations

$$\lambda f = \int_\Omega \lambda G(., \xi)\, Lf(\xi)\, d\xi, \qquad f \in F,$$

hold with

$$\lambda G(\cdot, \xi) \in C(\bar{\Omega}).$$

Then, if $I_0, ..., I_m$ are compact intervals in $\mathbf{R}$ and

$$U = \{f \in F: \lambda_i f \in I_i\,,\ i = 0, ..., m\},$$

we may state the multivariate analog of the previous theorem.

THEOREM 2.3. *Let* $\alpha = \inf\{\| Lf \|_{L^1[\Omega]}: f \in U\}$. *Then there are distinct points* $t_0, ..., t_r$ *in* $\bar{\Omega}$, $r \leqslant m$, *and real numbers* $a_0, ..., a_r$ *such that the multivariate generalized spline function*

$$s(x) = \sum_{j=0}^{r} a_j G(x, t_j)$$

satisfies $\lambda_i s \in I_i$, $i = 0, ..., m$ *and* $\sum_{j=0}^{r} | a_j | = \alpha$, *i.e.*, $\| Ls \|_{M[\Omega]} = \alpha$. s *solves an extended minimization problem as before and satisfies the relation* (2.4).

(III) *A Mathematical Programming Application*

Discrete spline functions were introduced by Mangasarian and Schumaker [10] as minimizing a general forward difference operator $L: \mathbf{R}^l \to \mathbf{R}^{l-n+1}$ subject to affine constraints in the l^p norm, $1 \leqslant p \leqslant \infty$. Methods of mathematical programming were employed in the existence theory to deduce the closure of certain sets. We shall show here that, in the special case $p = 1$, there is a solution s such that Ls has support confined to a subspace of $\mathbf{R}^l$ of dimension $m + 1$ if there are $m + 1$ constraint functionals.

Specifically, let L be of the form

$$(Lx)_j = \sum_{\nu=1}^{n} a_\nu x_{\nu+j-1}\,, \qquad j = 1, ..., l - n + 1, \tag{2.7}$$

where $a_n \neq 0$. Then L maps $\mathbf{R}^l$ onto $\mathbf{R}^{l-n+1}$ and, by the rank-nullity theorem, the null space N of L is of dimension $n-1$. The complement M of N in $\mathbf{R}^l$ is of dimension $l-n+1$ and L maps M bijectively onto $\mathbf{R}^{l-n+1}$. Thus, if $\Lambda = \{\lambda_0, ..., \lambda_m\}$ is a linearly independent set of linear functionals on $\mathbf{R}^l$ of the form

$$\lambda_i x = \sum_{j=1}^{l} a_{ij} x_j, \; x \in \mathbf{R}^l, \tag{2.8}$$

$i = 0, ..., m$, such that $N \subset \operatorname{span}_{0 \leqslant i \leqslant m} \{a_{i1}, ..., a_{il}\}$ and if $I_0, ..., I_m$ are compact intervals in $\mathbf{R}$ and

$$U = \{x \in \mathbf{R}^l : \lambda_i x \in I_i, \; i = 0, ..., m\}$$

then we have the following consequence of Theorem 1.1.

THEOREM 2.4. *Let* $\alpha = \inf\{\sum_{j=1}^{l-n+1} |(Lx)_j| : x = (x_1, ..., x_l) \in U\}$. *Then there exists a vector* $s \in U$ *satisfying* $\sum_{j=1}^{l-n+1} |(Ls)_j| = \alpha$ *and, moreover, the support of* Ls *is confined to at most* $m+1$ *components.*

Proof. Take X to be $l-n+1$ distinct points with the discrete topology.

3. APPROXIMATION

THEOREM 3.1. *Let* $f \in W^{n,1}(I)$, I *a compact interval in* $\mathbf{R}$, *with* $\| D^n f \|_{L^1(I)} \leqslant 1$ *and* $D^j f(a) = 0$ *for* $0 \leqslant j \leqslant n-1$. *Let a partition* $\Delta : a = x_0 < \cdots < x_m = b$ *of* I *be specified with* $m \geqslant n-1$ *such that* Δ *contains a fixed set* $\{Y_1, ..., Y_n\}$ *of* n *points. Let* s *be the spline function, guaranteed by Corollary 2.2, agreeing with* f *on* Δ. *Then there exists a positive constant* C, *independent of* f *and* Δ, *such that*

$$\| D^j(f - s) \|_{L^r(I)} \leqslant C h^{n-1-j}, \quad 0 \leqslant j \leqslant n-1, \qquad \text{for} \quad 1 \leqslant r \leqslant \infty. \tag{3.1}$$

Moreover, the order given by (3.1) *is sharp for* $n = 2$, $j = 0$, $r = 1$ *and* $I = [0, 1]$: *For each sufficiently small* $\epsilon > 0$,

$$\| \bar{\epsilon} x^{1+\epsilon} - s_m \|_{L^1(I)} \geqslant C_\epsilon \left(\frac{1}{m}\right)^{1+\epsilon}, \; C_\epsilon > 0, \tag{3.2}$$

where $s_m(i/m) = \bar{\epsilon}(i/m)^{1+\epsilon}$, $i = 0, ..., m$; *here* s_m *is the spline interpolant of Corollary* 2.2, $m = 1, 2, ...$, and $\bar{\epsilon} = (1+\epsilon)^{-1}$.

LEMMA 3.2. *Let* f, Δ *and* s *be given as in Theorem* 3.1. *There exists a positive constant* C', *independent of* f *and* Δ, *such that*

$$\| D^{n-1}(f - s) \|_{L^\infty} \leqslant C'.$$

Proof. Since,

$$f^{(n-1)}(x) = \int_a^x f^{(n)}(t)\,dt,$$

it follows that $\| D^{n-1}f \|_{L^\infty[I]} \leqslant 1$. Since

$$\| D^{n-1}s \| \leqslant | D^{n-1}s(a+)| + 1,$$

it suffices to show that $| D^{n-1}s(a+)|$ is bounded independently of f and Δ. Now s may be written

$$s(x) = P(x) + \sum_{j=0}^{r} a_j(x - t_j)_+^{n-1}, \tag{3.3}$$

where $\sum_{j=0}^{r} | a_j | \leqslant 1$, P is a polynomial of degree $n - 1$ and $a < t_0 < \cdots < t_r < b$.

Thus, $D^{n-1}s(a+) = D^{n-1}P(a)$. Now since the space $\mathscr{P}_{n-1}$ of real polynomials of degree $n - 1$ is of finite dimension n, it follows that any two norms are equivalent. In particular, any set of such polynomials bounded in the norm

$$\| P \|\, \mathscr{P}_{n-1} = \sum_{i=1}^{n} | P(Y_i)| \tag{3.4}$$

is bounded in the norm

$$||| P |||\, \mathscr{P}_{n-1} = \sum_{j=0}^{n-1} | c_j |, \quad P(x) = \sum_{j=0}^{n-1} c_j x^j. \tag{3.5}$$

We shall consider the set of polynomials P, given by (3.3), as f and Δ vary according to the hypotheses of Theorem 3.1. Now on the point set $\{Y_1, ..., Y_n\}$, f and s agree so that

$$P(Y) = f(Y) - \sum_{j=0}^{r} a_j(Y - t_j)_+^{n-1}, \tag{3.6}$$

if $Y \in \{Y_1, ..., Y_n\}$. Now the representation

$$f(x) = \int_a^b \{(x - t)_+^{n-1}/(n - 1)!\}\, f^{(n)}(t)\,dt$$

implies that $\| f \|_{C[a,b]} \leqslant (b - a)^n/n!$ and, clearly, the function

$$g(x) = \sum_{j=0}^{r} a_j(x - t_j)_+^{n-1}$$

satisfies

$$\| g \|_{C[a,b]} \leqslant (b - a)^{n-1},$$

so that the set of polynomials P, given by (3.3), is bounded in the norms (3.4) and (3.5). In particular, the coefficients c_{n-1} are uniformly bounded so that the numbers

$$D^{n-1}s(a+) = D^{n-1}P(a) = (n-1)!\, c_{n-1}$$

are uniformly bounded. This completes the proof of the lemma. Proof of Theorem 3.1: Standard arguments employing Rolle's Theorem (cf. [8, Eq. (2.4)]) give the inequalities

$$\| D^j(f - s)\|_{L^r(I)} \leqslant h(j+1) \| D^{j+1}(f - s)\|_{L^r(I)}, \qquad 0 \leqslant j \leqslant n-2. \tag{3.5}$$

Combining (3.5) with Lemma 3.2, we obtain (3.1) with $C = C'(j+1) \cdots (n-1)(b-a)^{1/r}$.

To obtain (3.2), we first observe that the knots $t_0, \ldots, t_r$ of s_m occur precisely at a subset of the interior nodal points $x_1, \ldots, x_{m-1}$. To show this, suppose first that no three adjacent graph points $(x_i, f(x_i))$ are collinear. Then, setting $J_i = [x_i, x_{i+1}]$, $i = 0, \ldots, m-1$, we see that, either $t_i \in J_i$, or $t_i \in J_{i+1}$. In either case, the magnitude of the difference in slopes of two linear functions, interpolating at the nodes contained in $J_i \cup J_{i+1}$ such that their graphs form a sawtooth on $J_i \cup J_{i+1}$ and meet at a point distinct from $(x_i, f(x_i))$, must strictly exceed the magnitude of the difference in slopes for the sawtooth function on $J_i \cup J_{i+1}$ with knot at x_i. The case of collinear graph points is similar. Having established that the knots of s_m are a subset of the nodes, (3.2) is a simple consequence of an adaptation of an argument first used by Schultz and Varga [11], as we now show.

Now fix $0 < \epsilon < 1$ and let

$$0 < \tau(\epsilon) = \inf_{P \in \mathscr{P}_1} \{ \| \bar{\epsilon} x^{1+\epsilon} - P \|_{L^1(0,1)} \}.$$

Then, for the interval $J_0 = [0, h]$, we have, after a change of variable $x = hy$,

$$\| \bar{\epsilon} x^{1+\epsilon} - s_m \|_{L^1(J_0)} \geqslant h^{2+\epsilon}\, \tau(\epsilon).$$

Precisely the same analysis yields

$$\| \bar{\epsilon} x^{1+\epsilon} - s_m \|_{L^1(J_i)} \geqslant h^{2+\epsilon}\, \tau(\epsilon), \qquad i = 1, \ldots, m-1$$

for $J_i = [ih, (i+1)h]$. Thus,

$$\| \bar{\epsilon} x^{1+\epsilon} - s_m \|_{L^1(0,1)} = \sum_{i=0}^{m-1} \| \bar{\epsilon} x^{1+\epsilon} - s_m \|_{L^1(J_i)} \geqslant m h^{2+\epsilon}\, \tau(\epsilon) = h^{1+\epsilon}\, \tau(\epsilon).$$

Now the function $f(x) = \bar{\epsilon}x^{1+\epsilon}$ satisfies $f \in W^{2,1}(0, 1)$ with $\|f''\|_{L^1[0,1]} = 1$ and $f(0) = f'(0) = 0$. Thus, choosing $Y_0 = 0$ and $Y_1 = 1$ we see that f and Δ satisfy the hypotheses of the first part of Theorem 3.1. This concludes the proof of the theorem.

Remark. The reader will observe that the order of approximation in (3.1), which is seen to be best possible in a generic sense for *this approximation process*, is of order one less than that achievable by optimal linear approximation processes (cf. the linear approximation process, defined for splines of degree $n - 1$, constructed in [15] and valid for $W^{n,1}$ functions). The explanation for this appears to be that the approximation process defined by Theorem 3.1 is actually intended for the larger class of functions whose nth derivatives, in the sense of measures, have variation (as measures) not exceeding one.

Acknowledgment

The authors acknowledge helpful comments from Prof. K. Scherer.

References

1. N. Dunford and J. T. Schwartz, "Linear Operators," Part I, Wiley, Interscience, New York, 1958.
2. S. D. Fisher and J. W. Jerome, The existence, characterization and essential uniqueness of solutions of L^∞ extremal problems, *Trans. Amer. Math. Soc.* **187** (1974), 391–404.
3. S. D. Fisher and J. W. Jerome, Elliptic variational problems in L^2 and L^∞, *Indiana J. Math.* **23** (1974), 685–698.
4. A. Gleason, "Fundamentals of Abstract Analysis," Addison-Wesley, Reading, MA, 1966.
5. M. Golomb, $H^{m,p}$-extensions by $H^{m,p}$-splines, *J. Approximation Theory* **5** (1972), 238–275.
6. R. B. Holmes, R-splines in Banach spaces. I, *J. Math. Anal. Appl.* **40** (1972), 574–593.
7. A. Ioffe and V. Tihomirov, Extension of variational problems, *Trans. Moscow Math. Soc.* **18** (1968), 207–273.
8. J. W. Jerome, Minimization problems and linear and nonlinear spline functions, II. Convergence, *SIAM J. Numer. Anal.* **10** (1973), 820–830.
9. O. L. Mangasarian and L. L. Schumaker, Splines via optimal control, *in* "Approximations with Special Emphasis on Spline Functions" (I. J. Schoenberg, Ed.), pp. 119–156, Academic Press, New York, 1969.
10. O. L. Mangasarian and L. L. Schumaker, Discrete splines via mathematical programming, *SIAM J. Control* **9** (1971), 174–183.
11. M. H. Schultz and R. S. Varga, L-splines, *Numer. Math.* **10** (1967), 345–369.
12. P. W. Smith, $W^{r,p}$-splines, Dissertation, Purdue University, Lafayette, IN, 1972.
13. K. Yosida, "Functional Analysis," Academic Press, New York, 1965.
14. J. Favard, Sur l'interpolation, *J. Math. Pures Appl.* **19** (1940), 281–306.
15. J. Jerome, On uniform approximation by certain generalized spline functions, *J. Approximation Theory* **7** (1973), 143–154.